A CASE STUDY INTO THE USE OF DYNAMIC GRAPHICAL SOFTWARE FOR THE LEARNING OF THE PROPERTIES OF QUADRATIC FUNCTIONS

Stephen Godwin, Rosamund Sutherland

Graduate School of Education, University of Bristol,

Rob Beswetherick, John Cabot City Technology College, Bristol

ABSTRACT

This paper draws from a large project whose overall aim is to examine the ways in which new technologies can be used effectively in educational settings to enhance learning. Its particular focus is within one of the mathematics design initiatives looking at the use of a graphing application to help with an understanding of quadratic functions. It contains an exploration of the ways in which the student, Kay, uses the software package Omnigraph, to work on quadratic equations supported by the teaching of Rob Beswetherick, one of the Mathematics teachers involved with the project. There is an analysis of the types of output she produces, and the nature of these outputs. Within this paper there is a consideration of the balance of prescriptive and experimental tasks and their role in scaffolding learning, a reflection on the role of the scaling facility of the software to change the appearance of functional representations and an assessment of the effectiveness of the learning environment for a particular student Kay. A balance of structured and scaffolded experimentation is suggested as being effective with respect to the learning of the student within the case study. The pre- and post-initiative tests and interviews given to the student provide evidence of the student's learning with respect to quadratic functions. Evidence is presented to suggest that the use of the scaling facility of the software has allowed Kay to understand that the graphs of functions can appear in a variety of different ways.

INTRODUCTION

This paper derives from a large ESRC funded Interactive Education project which is examining the ways that new technologies can be used in educational settings to enhance learning. The research reported here originates from the mathematics section of the Teaching and Learning strand of the project and discusses preliminary findings from one of several "design initiatives" conducted as a result of teacher/researcher partnerships.

ICT provides many opportunities for dynamic graphing ranging from using hand-held graphic calculators, using application software such as spreadsheets, and using specialist graphing software such as Omnigraph. There has been much reflection about their potential affordances within the literature and some debate as to whether they are cognitive tools that extend the ability of the individual, mathematical microworlds in which the student becomes immersed, or graphs becoming cognitively divorced from the functions that produce them. Hennesy et al (2001) in a study into the use of functions with graphing calculators lists ways in which graphing appears to be facilitated. These include being able to generate visual representations of algebraic functions, encouraging translations between different representations, speeding up the process of producing graphs, and working with and manipulating multiple graphs. They also go on to talk about the role of the technology in encouraging an "exploratory approach" (ibid).

Zaslavsky, Sela and Leron (2002) consider the limitations of graphs as representations of algebraic functions. In their study they consider the concept of slope and the conflict between slope as a mathematical notion, and slope as a visual feature of the plotted graph and how the changing of scale might create cognitive dissonance in the learner. The fact that graphing software is dynamic, and that scales can be changed, might create confusion for students but on the other hand it could be used to open up an understanding of the limits of representations. There is also the consideration that graphs are not merely representations but they are mathematical entities and concepts in their own right. David Pimm (1995) reflects on the use of computers and their effects on graphing. He considers how things might look different from unfamiliar points of view or perspectives. He states, "Take the case of a graph of a function. Whatever you see, you think you see all of "it", a single thing complete. What does a function really look like?" (p. 123). A potential value of graphing systems is their ability to change the "view" and hence provide a realisation that things might appear different but still be mappable onto the same mathematical object. This an important idea and could lead to investigations such as what features remain the same when you change a scale, and what features change.

Goldenberg (1991) considers the ways in which students experiment with dynamic geometry and considers the problem of unstructured experimentation. He states that "students do not, in general, seem to know how to perform meaningful experiments" (page 221). He lists the difficulties that they often have in their exploration. These include the changing of too many parameters at a time and hence the difficulty they might have in seeing the effects of modifying variables. Although within this article he is referring to dynamic geometry many of these difficulties could apply to the use of graphing environments. It is here that the teacher can play a vital role in structuring the tasks by allowing focused experimentation on some particular aspect. A continuum of experimentality could be considered ranging from giving a completed function for students to enter into the computer, through to experimentation on one part of a group of related functions (e.g. quadratics) through to total free reign and potential anarchy. It is interesting to consider what might be the best structures for allowing the students to experiment effectively and also to see the range of student behaviours and their potential understandings generated by their individual and interactive behaviours.

This paper focuses on one individual within the wider study and how they used Omnigraph within a series of four lessons investigating the properties of quadratic functions. The lessons contained a mixture of whole class teacher directed activities and individual student work with the computer graphing package Omnigraph. The teacher also composed a number of worksheets that the students worked with during the whole class activities and their individual work on the computers. Although students were working one to a machine there was considerable interaction with neighbouring students and the teacher.

KAY – A SHORT BACKGROUND

Kay is a year 9 student in Rob Beswetherick's class. Her pre-initiative attainment levels in Mathematics were said to be slightly below average for the class. She is fairly computer literate and comes from a household with a high degree of computer infusion. She said that her father was an IT manager and that they had four computers in her home. She is capable of installing her own software and her father encourages her to do this in order that she will learn. She uses the Internet to play games. Sometimes these are played on-line, although not directly with other people, and sometimes she will download the games. She has used the Internet for History, English (looking up things about Macbeth), and RE (looking up things about the holocaust) and also uses it for homework. She said that she had not used the Internet for Maths.

KAY'S PRE-INITIATIVE ASSESSMENT PLUS INTERVIEW REFLECTIONS RELATING TO THE ASSESSMENT

The initial assessment given to the students involved three questions where the students were asked to plot the graphs of three different quadratic functions. These were; $y = x^2$, $y = x^2 + 3$, and $y = (x - 4)^2$. Based on the initial written assessment there was no evidence that Kay was able to plot the given graphs and knew how to use the grids provided (see Fig. 1).During the interview it was clear that she knew that x^2 was x times x (not 2 times x for example), and demonstrated that she knew that 6^2 was 36. She referred to the graph origin as "nought" and realised that the plot of a function did not have to pass through the origin. There was also a further, more challenging question given to the students during the initial interview (see Fig. 2). The students were given some time to look at this question and then asked about it. Kay found the additional question given in the interview very tricky although she was able to determine the value of a mirrored coordinate C (see Fig. 2) once she knew the value of the coordinate B.

SEQUENCE OF LESSONS

This particular mathematics initiative consisted of four sessions of about 40 minutes duration. In the sessions the students used Omnigraph (a graphing software package) to investigate the behaviour and properties of quadratic equations. The teacher, Rob Beswetherick, had used Omnigraph with other classes but had never tried using a "hands on" approach with the students. Rob Beswetherick began the lessons with an introduction of approximately 5 to 10 minutes duration. In the first lesson Rob asked the students to initially familiarise themselves with the features of the Omnigraph software. This included entering graphs, and familiarising themselves with the zooming feature. This allowed the x and y scales to be adjusted simultaneously. In subsequent sessions Rob Beswetherick would highlight some feature of the quadratic equation and give an example of how it affected the plot of the equation. For example he might plot a graph, ask the students what the effect of changing a particular parameter in an equation might be and then he would plot the resultant graph. The worksheets that the students were given provided space for the filling in of their answers to some of the class questions. In one type of question Rob Beswetherick would project a graph on the screen and the students had to write down an equation for the graph. In the other type he would write one or more equations on the whiteboard and the students would have to sketch it in the appropriate part of their worksheets.

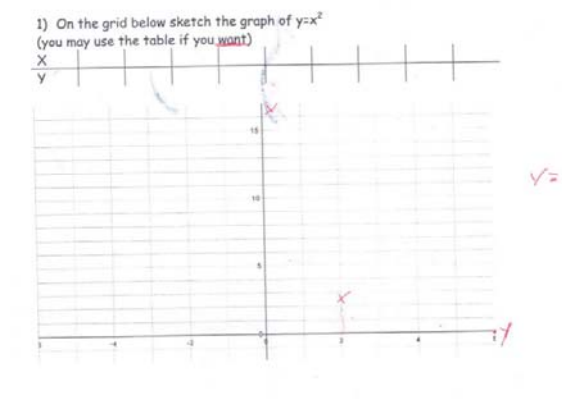
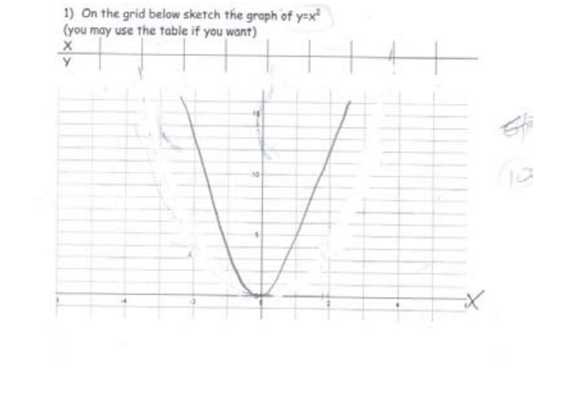
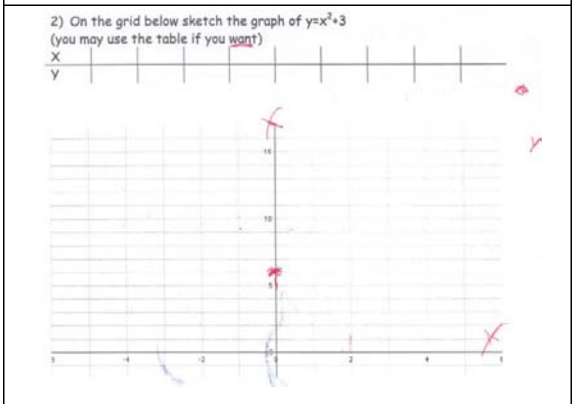
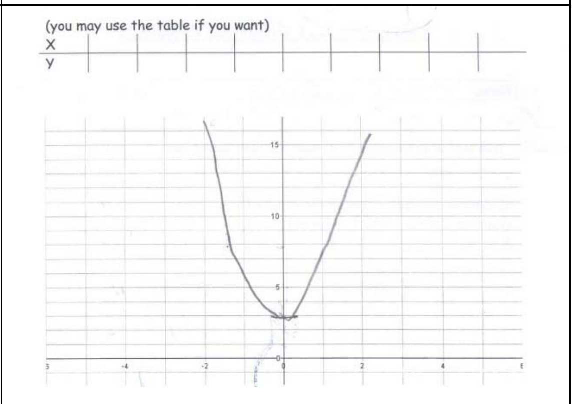
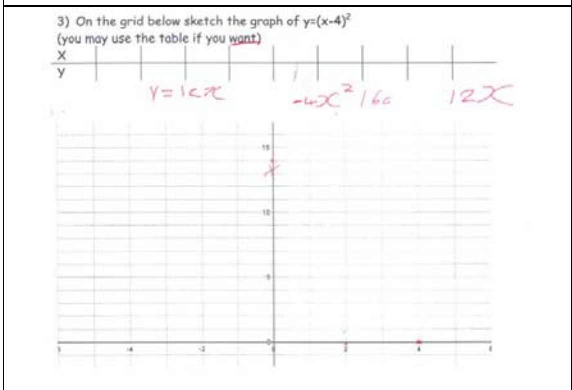
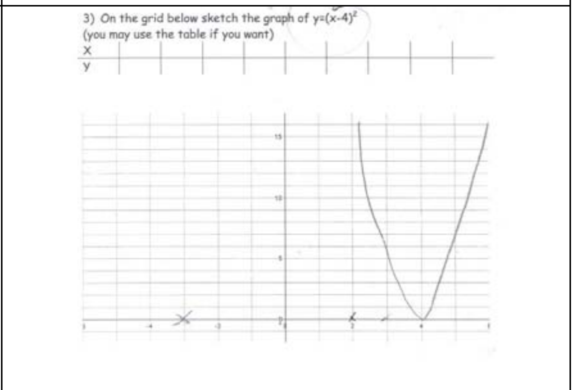
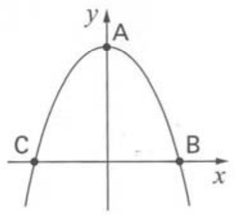
Pre-initiative assessment results	Post-initiative assessment results
	
	
	

Fig. 1: Pre-initiative and post-initiative assessment results for Kay

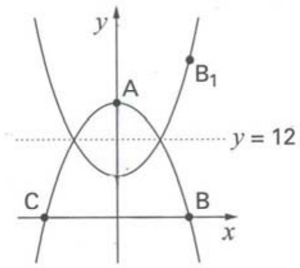
14. The diagram shows a sketch of the curve $y = 16 - x^2$



(a) What are the coordinates of points A, B and C?

A (,) B (,) C (,)

The curve $y = 16 - x^2$ is reflected in the line $y = 12$



(b) B_1 is the reflection of B

What are the coordinates of B_1?

B_1 (,)

(c) What is the equation of the new curve?

Fig. 2: Additional question given to the students for reflection during the initial interview and as part of the post-initiative assessment (Source: SATS question).

Following the introduction to the lesson the students would work on the computers and follow the tasks given in the worksheets. Although they were working alone, the students would often interact with their neighbours. This sometimes involved looking at another's work or verbal interaction. The teacher would also interact with the students. It was also clear that the students could hear student to student, and teacher to student interactions that were occurring in other areas of the classroom. It is, however, difficult to assess the impact of these on individual learning although it is felt to be important in the consideration of learning as a socially constructed activity. The tasks on the worksheet were directed towards analysing the effects of varying parts of generalised quadratic equations in a structured way. In some of the questions the graphs that the students were asked to plot were prescribed and in other questions the students were able to be more experimental.

These worksheets were filled in as the students worked on the activities. The activities used two basic structures:

$y = ax^2 + b$

$y = +/-(c + ax)^2 + b$

Although these were initially simplified (with a= 1 and b = 0 for example). Some of the functions on the worksheets that the students had to enter into the graph were wholly prescribed such as:

"What is the difference between $y=(x)^2$ and $y=-(x)^2$?"

Some questions were more open although a specific example was given e.g.

"What about if you have decimal numbers for 'a' or 'b'? Eg $y = -(x+0.5)^2+2.8$"

Some were very open ended e.g.

"What changes and what stays the same when you change the 'b' and 'c' numbers in $y = (x + c)^2 + b$?"

Another area of the graphics package that the students were encouraged to use was the zooming facility. This enabled the students to zoom in and out and thus effectively change the scale of the graph. The effect of zooming makes the graphs appear different and thus might provide students with an insight into the effects of scaling on graph appearance.

POST INITIATIVE ASSESSMENT AND REFLECTIONS RELATING TO THE ASSESSMENT

Immediately at the end of the series of lessons all the students were again given the assessment that they had completed at the beginning. This time the trickier question (see Fig. 2) was included. Several weeks after a group of students were re-interviewed. Kay and Jed were interviewed together on two occasions. No further work on the graphing of functions had been performed in class before these interviews. The first post-intitiative interview involved asking the students about what they felt about the activities, and also included a probing of the post-initiative test.

The second interview (performed with Jed and Kay together) involved the use of a laptop and included mini-tasks designed to demonstrate the students knowledge and skills.

INITIAL INVESTIGATION AND EMERGING THEMES

The initial investigation has involved an eyeballing of the data followed by a more detailed analysis of the graphical output of Kay during the sequence of lessons (described below). Although the focus of this particular paper is on Kay there will be a comparative analysis focusing on other individuals within the class and then broadening out.

Some current areas of interest that are emerging include:

- how the affordance of experimentation enables the students to visualise and understand the properties of quadratic equations

- how the teacher is important in the scaffolding of the tasks that help to focus the experimentation
- how the ease of which the x and y axis scales can be changed within the software might enable students to gain a greater insight into the effects of the scale on how the function appears within the graphical representation

Other emerging themes are not dealt with within this paper.

ANALYSIS

Analysing the graphic output

One of the ways of examining the degree of experimentation is to observe and record the types and sequence of graphs that Kay produced whilst working with Omnigraph when following the tasks on the worksheets. In all four sessions one of the video cameras was focused on Kay's work on the computer. After the initiative the graphs that Kay produced were sketched. Using the relevant worksheets and analysing the video each graph was initially numbered and then related to the part of the worksheet that Kay was currently engaged in. Where it was possible to do so the equation that had been used to generate the graph was ascertained (in some incidences, although the exact equation could not be known it was possible to get a close estimate). The graphs that Kay produced were then compared with the questions given in the worksheet and classified into three types. These were coded as follows:

A **Anchor graphs** – these are graphs that act as a kind of base point

e.g. $y = x$, $y = x^2$, or a return to a "starter" graph

P **Prescribed graphs** – graphs given by the teacher (usually on the worksheet)

e.g. plot the graph of $y = 3 + x^2$

Sometimes they are given as examples but the student uses the actual example to produce the graphical output

e.g. "What about negative numbers for 'a'? Eg $y = x^2 + 0.5$"

E **Experimental graphs**

These may be in the form of more open ended types of question

e.g. "What changes and what stays the same when you change the "a" number in $y = -(x + a)^2$?"

Or they might involve free experimentation that is not directly related to the worksheets.

Table 1 gives a summary of the number of such graphs produced in each of the sessions.

Runs

There were certain occasions where Kay produced a "run" of very closely related graphs that were not necessarily clearly prescribed by the worksheet. These "runs" raise questions in relation to how these series are created and in what ways they are related to the tasks on the worksheet. They might start from an initial prescribed graph followed by a series of related experimentals.

Changing the Scale using the Zooming Feature

There were several occasions where the zooming facility was used. In the first lesson it was used within the context of becoming familiar with the functionality of Omnigraph but subsequent to this it was largely used to adjust the graph to aid visualisation (either to see graphs that were 'off scale', or to 'better present' the function). A simple count has been made of these incidences and these are also shown in the summary table (see Table 1).

When analysing the graphs in conjunction with the worksheets it is clear that the worksheets play an important part in directing the nature of the experimentation of this particular student.

Kay's understanding of scaling

The first incidence of playing with the scaling occurred very early on in the first lesson. Another incidence of the change of scale was also demonstrated in the third lesson where scaling was necessary for Kay to be able to view the characteristic U shape of the graph (this time upside down). Here rather than merely having experimented with scaling she has used it to be able to get a better view of the function.

	Number of incidents of Anchor(A) graphs	Number of incidents of prescribed (P) graphs	Number of incidents of Experimental (E) graphs	Total number of graphs (observed)	Number of times zoom was used	Number of identified runs
Session 1	2	7+	13+	22+[1]	1+	3
Session 2	2	2	7	11+	0	2
Session 3	1	1	11	13+	4+	2/3
Session 4	1	17+	7	25+	0+	1
Totals	6	27+	38+	71+		

[1]There may be incidences of events that are not observed i.e. they may occur very quickly or when the camera was occasionally not on the computer screen

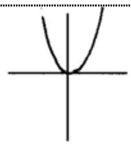
Table 1: Classification of Kay's graphing actions when using the graphing package on the computer

It is also interesting that before this scaling takes place the girls on her left had been repeatedly chanting “zoom, zoom, zoom”. In the post-initiative interview using the computer Kay clearly demonstrated an understanding of how changing the scaling affected the appearance of a quadratic when presented with sketches of extreme points of view (see Fig. 3).

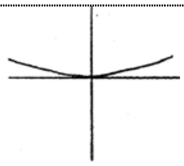
Test performance

In the pre-initiative testing Kay had not been able to produce any of the graphs by conventional plotting methods and did not appear to know how to do so when probed by the interviewer about it. In the post-initiative testing although Kay had not plotted the graphs she had fairly accurately sketched all three graphs (Figure 2) and demonstrated a knowledge of how changing the variable parameters of quadratics effected their positioning within the graphical representation. She had retained this knowledge a few weeks after the completion of the lessons. Further interview evidence supported a convincing understanding of the behaviour of quadratic graphs when the coefficients and constants were varied. In one question Kay and Jed were asked to enter the formula of $y = (x + 2)^2$ into the computer, given the sketch of the graphs shown below (see Fig. 4) and asked to reproduce them on the computer. They were able to do this together without help.

The two students in this interview (Kay and Jed) began by plotting the graph of $\mathbf{y = x^2}$ using the Omnigraph software.

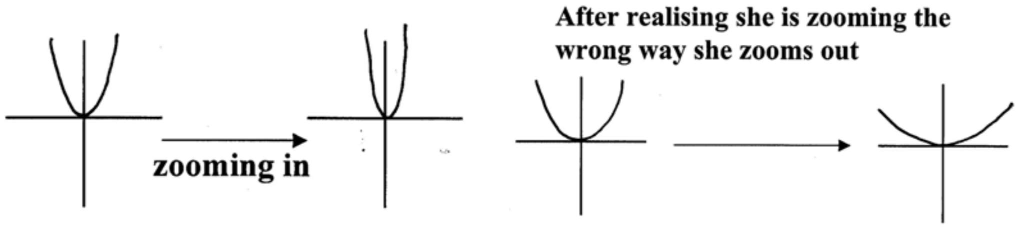


The researcher then asked the students to attempt to make the graph look like the sketch and yet "still be the same graph".

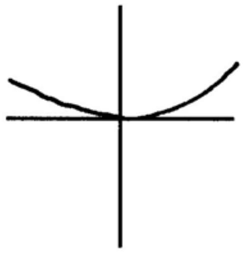


Kay asked whether the graph was to be made to look like the sketch without changing the equation. The researcher replied that she should not change the equation.

Kay started to zoom out going in the wrong direction but she quickly realised this and began to zoom in.



The researcher then prompted Kay to move in a little more which she quickly did.



It was clear that Kay realised that the appearance of the graph could be changed not only by using the zooming function but also by changing the equation.

Fig. 3: The Effect of Scaling on Graph Appearance

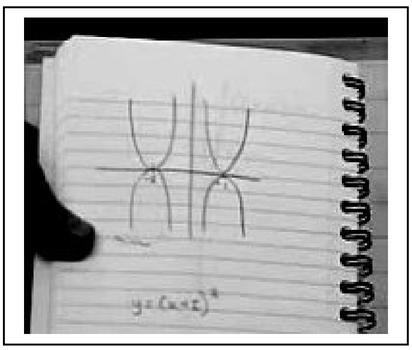


Fig. 4: The sketch given to the students that they were successfully able to replicate completed on the computer in the final interview

CONCLUSIONS

There was clear evidence that Kay enjoyed working with the software and was actively engaged in the tasks. This was reflected by her eager participation in whole class interactions, a praising comment to the teacher after one of the sessions, a positive post test remark, and her understanding demonstrated in the post-initiative interviews. Working with the package allowed Kay to plot at least 71 graphs of which 38 were of an experimental nature. These experiments were scaffolded by the teacher who designed the tasks so that the students were directed to experiment with certain parts of the equation. This helps and enables the students to keep all the variables that make a quadratic fixed apart from one, and thus helps the students to more clearly see its effect on the representation. If the students were not provided with this structure it might make it far more difficult to see the effects of the individual parts since changing the variables together might mask their individual effects. The structuring of the experimentation by the teacher helps to address the problem that Goldenburg (ibid) talked about with respect to the fact that he felt that students found it difficult to perform meaningful experiments. There is always the danger that the software may allow a lot of random playing without thought or reflection about what is being done but by structuring the play and providing tasks for the students their learning can be directed. It was interesting to note that at the end of the final session Kay began to explore $y = ax^2 +bx +c$. This had not been suggested in the tasks.

Graphing software also allows the easy changing of scales that are not afforded by traditional methods of graph plotting. This means that students do not always see a homogenous representation of a mathematical function. The author believes that in this case the students learn an important point about the representation of mathematical ideas and functions. It was clear from the research that Kay understood that a quadratic equation did not have to look like a U, that it could be made to look like a horizontal line or a tight V. She showed a deftness in being able to control the scaling to get the appropriate representation. With this particular software zooming in and out was done equally with both the x and y axis. Further questions could be asked and explored with zooming in the direction of one axis only, or using different types of scaling. Graphs are representations of functions which are in their turn either representations of things in the real world or abstract concepts but there should also be consideration of graphs as concepts in their own right. Perhaps the software allows the students to learn something more about the ways that things might be represented within mathematics and also the common features of quadratic functions.

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